

MATHEMATICS

ON THE TOPOLOGY OF THE SPACE OF ALL HOLOMORPHIC FUNCTIONS ON A GIVEN OPEN SUBSET ¹⁾

BY

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Let E and F be two complex Banach spaces, $\mathcal{P}(^m E; F)$ the Banach space of all continuous m -homogeneous polynomials from E to F ($m = 0, 1, \dots$), U a non-void open subset of E , $\mathcal{H}(U; F)$ the vector space of all holomorphic F -valued functions on U , and \hat{d}^m the differentiation mapping from $\mathcal{H}(U; F)$ into $\mathcal{H}(U; \mathcal{P}(^m E; F))$ of order $m = 0, 1, \dots$

A seminorm p on $\mathcal{H}(U; F)$ is said to be ported by a compact subset K of U if to every open subset V of U containing K there corresponds a real number $c(V) > 0$ such that

$$p(f) \leq c(V) \cdot \sup_{x \in V} \|f(x)\|$$

for every $f \in \mathcal{H}(U; F)$. The topology \mathcal{T}_ω on $\mathcal{H}(U; F)$ is defined by all such seminorms ported by compact subsets of U . Each of the following conditions is necessary and sufficient for p to be ported by K :

(1) Corresponding to every real number $\varepsilon > 0$ there is a real number $c(\varepsilon) > 0$ such that, for every $f \in \mathcal{H}(U; F)$,

$$p(f) \leq c(\varepsilon) \sum_{m=0}^{\infty} \varepsilon^m \sup_{x \in K} \left\| \frac{1}{m!} \hat{d}^m f(x) \right\|.$$

(2) Corresponding to every real number $\varepsilon > 0$ and open subset V of U containing K , there is a real number $c(\varepsilon, V) > 0$ such that, for every $f \in \mathcal{H}(U; F)$,

$$p(f) \leq c(\varepsilon, V) \sum_{m=0}^{\infty} \varepsilon^m \sup_{x \in V} \left\| \frac{1}{m!} \hat{d}^m f(x) \right\|.$$

U is ξ -equilibrated, where $\xi \in U$, if $(1-\lambda)\xi + \lambda x \in U$ for any $x \in U$, $\lambda \in \mathbf{C}$, $|\lambda| \leq 1$. If U is ξ -equilibrated, the Taylor series at ξ of any $f \in \mathcal{H}(U; F)$ converges to f in the sense of \mathcal{T}_ω . If U is ξ -equilibrated, the following condition is necessary and sufficient for p to be ported by K : corresponding to every open subset V of U containing K , there is a real number $c(V) > 0$

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such that, for every $f \in \mathcal{H}(U; F)$,

$$p(f) \leq c(V) \cdot \sum_{m=0}^{\infty} \sup_{x \in V} \left\| \frac{1}{m!} \hat{d}^m f(\xi) \cdot (x - \xi) \right\|.$$

The compact-open topology on the vector space $\mathcal{C}(U; F)$ of all continuous F -valued functions on U induces a topology \mathcal{T}_0 on $\mathcal{H}(U; F)$. We have $\mathcal{T}_0 \subset \mathcal{T}_\omega$; $\mathcal{T}_0 = \mathcal{T}_\omega$ if and only if $\dim E < \infty$, or $F = 0$. Each \hat{d}^m is continuous for the corresponding topologies \mathcal{T}_ω ; continuity of \hat{d}^m for some $m \geq 1$ and the corresponding topologies \mathcal{T}_0 requires $\dim E < \infty$, or $F = 0$. However a subset \mathcal{X} of $\mathcal{H}(U; F)$ is bounded for \mathcal{T}_ω if and only if it is bounded for \mathcal{T}_0 . Each of the following conditions is necessary and sufficient for \mathcal{X} to be bounded for \mathcal{T}_ω :

(1) Corresponding to every compact subset K of U , there is a real number $C \geq 0$ such that $\|f(x)\| \leq C$ for every $f \in \mathcal{X}$ and $x \in K$.

(2) Corresponding to every compact subset K of U , there are a real number $C \geq 0$ and an open subset V of U containing K such that $\|f(x)\| \leq C$ for every $f \in \mathcal{X}$ and $x \in V$.

(1') Corresponding to every $\xi \in U$, there are real numbers $C \geq 0$ and $c \geq 0$ such that, for every $m = 0, 1, \dots$ and $f \in \mathcal{X}$,

$$\left\| \frac{1}{m!} \hat{d}^m f(\xi) \right\| \leq C \cdot c^m.$$

(2') Corresponding to every compact subset K of U , there are real numbers $C \geq 0$ and $c \geq 0$ such that, for every $m = 0, 1, \dots$, $f \in \mathcal{X}$ and $x \in K$,

$$\left\| \frac{1}{m!} \hat{d}^m f(x) \right\| \leq C \cdot c^m.$$

(3') Corresponding to every compact subset K of U , there are real numbers $C \geq 0$ and $c \geq 0$, and an open subset V of U containing K , such that, for every $m = 0, 1, \dots$, $f \in \mathcal{X}$ and $x \in V$,

$$\left\| \frac{1}{m!} \hat{d}^m f(x) \right\| \leq C \cdot c^m.$$

Let $X \subset U$ be fixed, and suppose X meets every connected component of U . Then \mathcal{X} is bounded for \mathcal{T}_ω if and only if \mathcal{X} is equicontinuous on U and $\sup \{\|f(x)\| \mid f \in \mathcal{X}\} < \infty$ for every $x \in X$. Denote by $\mathcal{T}_{\infty, X}$ the topology on $\mathcal{H}(U; F)$ defined by the family of seminorms $f \rightarrow \|\hat{d}^m f(x)\|$ for $m = 0, 1, \dots$ and $x \in X$. If \mathcal{X} is \mathcal{T}_ω -bounded, \mathcal{T}_ω and $\mathcal{T}_{\infty, X}$ induce the same topology on \mathcal{X} ; also the uniform structures associated with \mathcal{T}_ω and $\mathcal{T}_{\infty, X}$ induce the same uniform structure on \mathcal{X} . If $f, f_l \in \mathcal{H}(U; F)$ for $l = 0, 1, \dots$, then $f_l \rightarrow f$ in the sense of \mathcal{T}_ω as $l \rightarrow \infty$ if and only if $\{f_l\}$ is \mathcal{T}_ω -bounded and $\hat{d}^m f_l(x) \rightarrow \hat{d}^m f(x)$ in $\mathcal{P}({}^m E; F)$ as $l \rightarrow \infty$ for every

$m=0, 1, \dots$ and $x \in X$. Also \mathcal{X} is \mathcal{T}_ω -relatively compact if and only if \mathcal{X} is \mathcal{T}_ω -bounded and $\{d^m f(x) | f \in \mathcal{X}\}$ is relatively compact in $\mathcal{P}({}^m E; F)$ for every $m=0, 1, \dots$ and $x \in X$.

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